# Bi-invariant metrics on groups

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# Overview

- Definition.
- Riemannian Geometry.
- Hamiltonian dynamics.
- Group Theory.
- Biology.
- General outlook.
- Free groups.

## *"It is impossible to understand an unmotivated definition. . . "*

### Definition

Let $G$ be a group. A metric $d$ on $G$ is called ***bi-invariant*** if both the multiplication from the right and from the left are isometries:

$$d(xg, yg) = d(x, y) = d(gx, gy)$$

for all $x, y, g \in G$.

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- $\|g\| = d(g, 1)$ – a conjugation-invariant norm.
- $d(g, h) = \|gh^{-1}\| = \|g^{-1}h\|$.

# Riemannian geometry

- $G$ a Lie group with Lie algebra $\mathfrak{g}$.
- Choose an inner product on $\mathfrak{g} = T_1 G$.
- Propagate over $TG$ with left multiplication:

$$\langle X, Y \rangle_g = \langle dL_{g^{-1}} X, dL_{g^{-1}} X \rangle_1.$$

$\Longrightarrow$ Left-invariant metric on $G$.

# Riemannian geometry

$G = \mathrm{SO}(3)$
$\mathrm{Ad}\colon \mathrm{SO}(3) \to \mathrm{Aut}(\mathrm{so}(3))$

$G = \mathrm{SL}(2, \mathbf{R})$
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# Hamiltonian dynamics

- $(M, \omega)$ - a symplectic manifold.

$\Longleftrightarrow$ $\omega$ - closed non-degenerated two-form

$\Longrightarrow$ Isomorphism between vector fields and one-forms:

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| | | | | | |
|---|---|---|---|---|---|
| Group | $\mathrm{Diff}(M)$ | $\supseteq$ | $\mathrm{Symp}(M, \omega)$ | $\supseteq$ | $\mathrm{Ham}(M, \omega)$ |
| Algebra | $X$ | | $L_X \omega = 0$ | | $\iota_X \omega = dH$ |

# Hofer's metric

$$\|\psi\| = \inf_H \int_0^1 \operatorname{osc} H(t, -)\ dt$$

All known examples have infinite Hofer diameter.

# Autonomous metric

- $\psi \in \mathrm{Ham}(M, \omega)$.
- $\psi = \psi_1$, where $\{\psi_t\} \in \mathrm{Ham}(M, \omega)$.
- $\psi_t \quad \longleftrightarrow \quad X_t \quad \longleftrightarrow \quad H_t$.
- $\psi$ is **autonomous** if $H_t = H$ is time independent.

$$\|\psi\| = \min\{n \in \mathbf{N} \mid \psi = \alpha_1 \cdots \alpha_n,\ \alpha_i \text{ is autonomous}\}$$

# Group theory

$G$ a group generated by $S \subseteq G$; $S = S^{-1}$. Word norm and metric:

$$\|g\|_S = \min\{n \in \mathbf{N} \mid g = s_1 \cdots s_n,\ s_i \in S\}$$

$$d_S(g,h) = \|gh^{-1}\|_S$$

`right-invariant`

If $g^{-1}Sg = S$ for every $g \in G$ then the norm is conjugation-invariant and the metric bi-invariant.

# Examples

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  - $S$ – the set of all reflections.
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- Commutator subgroup $[G, G] \subseteq G$.
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  - Good topological interpretation.
- $\mathbf{F}_2 = \langle a, b \rangle$ - free group on two generators.
  - $S$ – all conjugates of $a$, $b$ and their inverses.
  - $d_S$ – ***bi-invariant***.
  - Good algorithms for computations (we have a software [2013]).

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- $(M, g)$ - Riemannian manifold.
  - $S \subseteq \pi_1(M, x)$ – the set of all closed geodesics.
  - Assume $S$ generates $\pi_1(M, x)$.

$\Longrightarrow$ the closed geodesic metric on $\pi_1(M, x)$.

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- Biologists found our algorithm in 1980!

# Bi-invariant word metrics

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- $G$ – normally finitely generated group.
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- $d_S$ – **the** bi-invariant word metric.
- Q: Is the diameter of $d_S$ finite or infinite?
  - $\mathbf{F}_n$ – infinite. `Non-el. hyperbolic`
  - $\mathrm{SL}(n, \mathbf{Z})$ – finite if $n \geq 3$. `Chevalley`
  - $\mathrm{SO}(5, \mathbf{Z}[1/5])$ – ? `Cocompact lattice`
  - $\mathrm{Diff}_0(S^n)$ – finite.
  - $\mathrm{Diff}_0(\Sigma_g)$ – infinite if $g \geq 2$.
  - $\mathrm{Diff}_0(S^n, \mathrm{vol})$ - ?, $n \geq 3$.
  - Etc. Ask questions at the end.

# The free group

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- $\|g^n\| = 4, 4, 6, 8, 10, 10, 12, 14, 16, 16, 18, 20, 22, 22, 24, 26, \cdots$
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- $\|[a,b]^n\| = 2, 4, 4, 6, 6, 8, 8, \cdots = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even} \end{cases}$
- In fact...

# Jaspars's Theorem

Let $g \in \mathbf{F}_n$ be any element in a free group. Then the sequence $\|g^n\|$ is *uniformly semi-arithmetic*. In particular, the limit

$$\lim_{n\to\infty} \frac{\|g^n\|}{n}$$

is rational.

# THANK YOU! Any questions?